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LETTER TO THE EDITOR

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Abstract. The Schwartz inequality is used to obtain some generalized uncertainty relations among higher-order moments of the position and momentum operators.

It is well known that one essential feature of quantum mechanics lies in the commutation relation [1]

$$[\hat{q}, \hat{p}] = \mathrm{i}\hbar \tag{1}$$

where $\hat{q}$ and $\hat{p}$ are the self-adjoint position and momentum operators respectively, and $\hbar = \frac{h}{2\pi}$ where h is the Planck constant. Any pair of Hermitian operators $(\hat{A}, \hat{B})$ satisfies the Schwartz inequality

$$\langle\hat{A}^2\rangle\langle\hat{B}^2\rangle \geqslant |\langle\tfrac{1}{2}\{\hat{A}, \hat{B}\}_+\rangle|^2 + |\langle\tfrac{1}{2}[\hat{A}, \hat{B}]_-\rangle|^2 \tag{2}$$

where the average is taken with respect to a state $|\psi\rangle$. Here $\{\ \}_+$ and $[\ ]_-$ stand for the anti-commutator and the commutator, respectively. The inequality in (2) becomes the equality for the states for which

$$\hat{A}|\psi\rangle = \lambda\hat{B}|\psi\rangle \tag{3}$$

where λ is a constant. Since

$$\Delta\hat{q} \equiv \hat{q} - \langle\hat{q}\rangle \qquad \Delta\hat{p} \equiv \hat{p} - \langle\hat{p}\rangle \tag{4}$$

satisfy the same commutation relation (1), we have for $\hat{A} = \Delta\hat{q}$ and $\hat{B} = \Delta\hat{p}$, the familiar Robertson–Schrödinger uncertainty relation [2]

$$\det\begin{vmatrix} \langle(\Delta\hat{q})^2\rangle & \langle\frac{1}{2}\{\Delta\hat{q}, \Delta\hat{p}\}_+\rangle \\ \langle\frac{1}{2}\{\Delta\hat{q}, \Delta\hat{p}\}_+\rangle & \langle(\Delta\hat{p})^2\rangle \end{vmatrix} \geqslant \frac{\hbar^2}{4}. \tag{5}$$

As $\langle\frac{1}{2}\{\Delta\hat{q}, \Delta\hat{p}\}_+\rangle$ has a continuous spectrum from $-\infty$ to $+\infty$, we arrive at the weaker Heisenberg uncertainty relation [3]

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geqslant \frac{\hbar^2}{4}. \tag{6}$$

In this letter, we present a class of uncertainty relations among some higher-order moments of $\hat{q}$ and $\hat{p}$ which follow from the Schwartz inequality.

Defining the operator

$$\hat{N} \equiv \mathrm{i}\hat{q}\hat{p} \tag{7}$$

it is easily verified using (1) that

$$[\hat{N}, \hat{q}] = \hat{q} \qquad [\hat{p}, \hat{N}] = \hat{p}. \tag{8}$$

A simple induction yields the relations

$$\begin{aligned} \hat{p}^n \hat{q}^n &= (-\mathrm{i}\hbar)^n (\hat{N} - n + 1)_n = (\hat{X} - a)(\hat{X} - 3a)\ldots[\hat{X} - (2n-1)a] \\ \hat{q}^n \hat{p}^n &= (-\mathrm{i}\hbar)^n (\hat{N} + 1)_n = (\hat{X} + a)(\hat{X} + 3a)\ldots[\hat{X} + (2n-1)a] \end{aligned} \tag{9}$$

where $(x)_n = x(x+1)\ldots(x+n-1)$, $n =$ positive integers, stands for the Pochhammer symbol, $a = \frac{1}{2}\mathrm{i}\hbar$, and the operator

$$\hat{X} \equiv \frac{\hat{p}\hat{q} + \hat{q}\hat{p}}{2} = -\mathrm{i}(\hat{N} + \tfrac{1}{2})\hbar. \tag{10}$$

Since $\hat{X}$ has a continuous spectrum from $-\infty$ to $+\infty$, it follows that

$$|\langle \hat{X}^m \rangle|^2 \geqslant 0 \qquad \text{for any integer } m. \tag{11}$$

Consequently, we find that

$$\begin{aligned} |\langle \tfrac{1}{2}\{\hat{q}^n, \hat{p}^n\}_+ \rangle|^2 &= |\langle \tfrac{1}{2}\{(\hat{X}+a)(\hat{X}+3a)\ldots(\hat{X}+(2n-1)a) \\ &\quad + (\hat{X}-a)(\hat{X}-3a)\ldots(\hat{X}-(2n-1)a\}\rangle|^2 \\ &\geqslant \begin{cases} \left(\dfrac{\hbar}{2}\right)^{2n} \{1 \times 3 \times 5 \times \cdots (2n-1)\}^2 & \text{for} \quad n \text{ even} \\ 0 & \text{for} \quad n \text{ odd.} \end{cases} \end{aligned} \tag{12}$$

Similarly,

$$|\langle \tfrac{1}{2}[\hat{q}^n, \hat{p}^n]_- \rangle|^2 \geqslant \begin{cases} \left(\dfrac{\hbar}{2}\right)^{2n} \{1 \times 3 \times 5 \times \cdots (2n-1)\}^2 & \text{for} \quad n \text{ odd} \\ 0 & \text{for} \quad n \text{ even.} \end{cases} \tag{13}$$

Thus, the Schwartz inequality gives (for all n)

$$\langle \hat{q}^{2n} \rangle \langle \hat{p}^{2n} \rangle \geqslant \left(\frac{\hbar}{2}\right)^{2n} \{(2n-1)!!\}^2 \tag{14}$$

where $(2n-1)!! = 1 \times 3 \times 5 \times \cdots \times (2n-1)$. We recognize that for $n = 1, 2$ (14) gives

$$\langle \hat{q}^2 \rangle \langle \hat{p}^2 \rangle \geqslant \frac{\hbar^2}{4} \qquad \langle \hat{q}^4 \rangle \langle \hat{p}^4 \rangle \geqslant \frac{9\hbar^4}{16}. \tag{15}$$

Equation (15) has recently been shown to follow from $sp(2R)$ invariance [4]. $sp(2R)$, the symplectic group in two dimensions, is the group of all linear canonical transformations that leaves the basic commutation relation (1) invariant. Since (14) holds for $(\Delta\hat{q})$ and $(\Delta\hat{p})$, we get

$$\langle (\Delta\hat{q})^{2n} \rangle \langle (\Delta\hat{p})^{2n} \rangle \geqslant \left(\frac{\hbar}{2}\right)^{2n} \{(2n-1)!!\}^2. \tag{16}$$

Equation (16) reveals that the higher-order moments are progressively weaker correlated, in view of the higher powers in $\hbar$.

With the advent of new techniques in quantum optics (see [5, 6] for instance), it should be possible to test these higher-order uncertainty relations in experiments. They will provide further evidence for the validity of quantum theory.

The article of Professor E C G Sudarshan started my interest in the subject and I thank him for discussions. I thank Mr Seetharaman Santhanam for typing this report.

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